Manifolds with Quadratic Curvature Decay and Slow Volume Growth

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To Detlef Gromoll on his 60th birthday

Abstract

We show that there are topological obstructions for a noncompact manifold to admit a Riemannian metric with quadratic curvature decay and a volume growth which is slower than that of the Euclidean space of the same dimension.

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1 Introduction

A major theme is Riemannian geometry is the relationship between curvature and topology. For compact manifolds, one can constrain the curvature and diameter and ask whether one obtains topological restrictions on the manifold. If the manifold is noncompact then a replacement for a diameter bound is a constraint on how the curvature behaves in terms of the distance from a basepoint. More precisely, let M be a complete connected n-dimensional Riemannian manifold. Fix a basepoint $m_0 \in M$.

Definition 1.1 *M has quadratic curvature decay (with constant $C > 0$) if for all $m \in M$ and all 2-planes P in $T_m M$, the sectional curvature $K(P)$ of P satisfies*

$$|K(P)| \leq C/d(m_0, m)^2. \tag{1}$$

Note that condition (1) is scale-invariant in that it is unchanged by a constant rescaling of the Riemannian metric. Many interesting results have been obtained when the sectional curvature has some faster-than-quadratic curvature decay [1, 8]. In this paper we concentrate instead on the case of quadratic curvature decay. In itself, condition (1) does not put any restrictions on the topology of a manifold. One can show that any connected smooth paracompact manifold has a Riemannian metric with quadratic curvature decay; see [9, p. 96] or Lemma 2.1 below. On the other hand, we will show that if in addition one restricts the volume growth of the metric, then one does obtain topological restrictions on M. The first question is whether M has finite topological type.

Definition 1.2 *M has finite topological type if M is homotopy-equivalent to a finite CW-complex.*

Definition 1.3 *M has lower quadratic curvature decay (with constant $C > 0$) if for all $m \in M$ and all 2-planes P in $T_m M$, the sectional curvature $K(P)$ of P satisfies*

$$K(P) \geq -C/d(m_0, m)^2. \tag{2}$$

Let B_t denote the metric ball of radius t around m_0 and let S_t denote the distance sphere of radius t around m_0. If M has lower quadratic curvature decay then by a standard argument, one can show that M has at most polynomial volume growth; see Lemma 3.1 below.

Proposition 1.1 *Suppose that M has lower quadratic curvature decay. If* $\mathrm{vol}(B_t) = o(t^2)$ *as* $t \to \infty$ *and* M *does not collapse at infinity, i.e.* $\inf_{x \in M} \mathrm{vol}(B_1(x)) > 0$, *then M has finite topological type.*

The $o(t^2)$ bound in Proposition 1.1 cannot be improved to $O(t^2)$, as shown in Example 3 below.

Next, we consider manifolds with volume growth slower than that of the Euclidean space of the same dimension.

Definition 1.4 *M has slow volume growth if*

$$\liminf_{t \to \infty} \mathrm{vol}(B_t)/t^n \ = 0. \tag{3}$$

There is a notion of an *end* E of M and of E being contained in an open set $\mathcal{O} \subset M$; see, for example, [2, p. 80].

Definition 1.5 *An end E of M is* tame *if it is contained in an open set diffeomorphic to* $(0, \infty) \times X$ *for some smooth connected closed manifold X.*

We remark that X is determined by E only up to h-cobordism. Hereafter we assume that M is oriented.

Proposition 1.2 *Suppose that M has quadratic curvature decay and slow volume growth. Let E be a tame end of M as in Definition 1.5. Then for any product* $\prod_k p_{i_k}(TX)$ *of Pontryagin classes of X and any bounded cohomology class* $\omega \in \mathrm{H}^l(X; \mathbb{R})$ *with* $l + 4\sum_k i_k = n - 1$,

$$\int_X \omega \cup \prod_k p_{i_k}(TX) = 0. \tag{4}$$

Example : There is no metric of quadratic curvature decay and slow volume growth on $\mathbb{R} \ \times \mathbb{C}\mathrm{P}^{2k}$.

Next, we give a sufficient condition for M to have a metric of quadratic curvature decay and slow volume growth.

Proposition 1.3 *Let X be a closed manifold with a polarized F-structure [5]. Suppose that $X = \partial N$ for some smooth compact manifold N. Then there is a complete Riemannian metric on* $M = \mathrm{Int}(N)$ *of quadratic curvature decay and slow volume growth.*

It follows from Proposition 1.3 that when n is even, there is a metric on $\mathbb{R}^n$ of quadratic curvature decay and slow volume growth. The case when n is odd is less obvious.

Proposition 1.4 *For all $n > 1$, there is a complete Riemannian metric on $\mathbb{R}^n$ of quadratic curvature decay and slow volume growth.*

If X is a closed oriented manifold with a polarized F-structure then the Pontryagin numbers and Euler characteristic of X vanish. Based on Proposition 1.3, one may think that under the hypotheses of Proposition 1.2, one could also show that the Euler characteristic of X vanishes. However, Proposition 1.4 shows that this is not the case, as the Euler characteristic of S^{n-1} is two if n is odd.

We can combine Propositions 1.2-1.4 to obtain some low-dimensional results.

Corollary 1.1 *Let N be a smooth compact connected oriented manifold-with-boundary of dimension n.*
1. If $n = 2$ then $\mathrm{Int}(N)$ has a metric of quadratic curvature decay and slow volume growth.
2. If $n = 3$ then $\mathrm{Int}(N)$ has a metric of quadratic curvature decay and slow volume growth if and only if ∂N consists of 2-spheres and 2-tori.
2. If $n = 4$, suppose that Thurston's Geometrization Conjecture holds. Then $\mathrm{Int}(N)$ has a metric of quadratic curvature decay and slow volume growth if and only if the connected components of ∂N are graph manifolds.

Finally, as in [4], there is an integrality result for the integral of the Gauss-Bonnet-Chern form, which we state without proof.

Proposition 1.5 *Suppose that M has a complete Riemannian metric g of quadratic curvature decay with $\mathrm{vol}(B_t) = o(t^n)$ and $\int_1^\infty \frac{\mathrm{vol}(B_t)}{t^n} \frac{dt}{t} < \infty$. Let $e(M,g) \in \Omega^n(M)$ be the Gauss-Bonnet-Chern form. Then $\int_M e(M,g) \in \mathbb{Z}$.*

We thank M. Gromov for pointing out the relevance of bounded cohomology.

2 Examples

1. Let N be a smooth compact connected n-dimensional manifold-with-boundary. Let h be a metric on ∂N. Given $c \geq 1$, consider the metric on $[1, \infty) \times \partial N$ given by $dt^2 + t^{2c} h$. Extend this to a smooth metric g on $\mathrm{Int}(N) = N \cup_{\partial N} [1, \infty) \times \partial N$. Then g has quadratic curvature decay and polynomial volume growth. By choosing c large, the degree of volume growth can be made arbitrarily large. Taking $c = 1$, we see that having quadratic curvature decay and volume growth of order $O(t^n)$ in no way restricts the topology of the ends.

2. For $c \in \mathbb{R}$, consider the metric on $[1, \infty) \times S^1$ given by $dt^2 + t^{2c} d\theta^2$. Cap this off by a disk at $\{1\} \times S^1$ to obtain a smooth metric g on $\mathbb{R}^2$. Then g has quadratic curvature decay. If $c < -1$ then $(\mathbb{R}^2, g)$ has finite volume. Hence the assumption of quadratic curvature decay gives no nontrivial lower bound on volume growth.

3. Start with the Euclidean metric on the annulus $A = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$. Add a handle to $\mathrm{Int}(A)$, keeping the metric the same near ∂A. Let h denote the corresponding metric on $T^2 - D^2$. With an obvious notation, for $j \in \mathbb{N}$, let $2^j \cdot (T^2 - D^2)$ denote the rescaled metric. Consider $(T^2 - D^2) \cup_{S^1} 2 \cdot (T^2 - D^2) \cup_{S^1} 4 \cdot (T^2 - D^2) \cup_{S^1} \ldots$ with its corresponding metric. Cap it off with a disk to obtain a smooth metric g_Σ on an infinite genus surface Σ. For $n \geq 2$, let $g_{T^{n-2}}$ be a flat metric on the $(n-2)$-torus. Then the product metric $(\Sigma, g_\Sigma) \times (T^{n-2}, g_{T^{n-2}})$ has quadratic curvature decay, volume growth of order t^2 and infinite topological type. This shows that the $o(t^2)$ condition in Proposition 1.1 cannot be improved to $O(t^2)$.

Lemma 2.1 *If M is a smooth connected paracompact manifold then M admits a complete Riemannian metric of quadratic curvature decay.*

Proof: First, M admits a complete Riemannian metric h of bounded sectional curvature [7]. Given $\phi \in C^\infty(M)$, put $g = e^{2\phi} h$. We have

$$R^i_{\ jkl}(g) = R^i_{\ jkl}(h) - \widetilde{\phi}^i_{\ k} h_{jl} + \widetilde{\phi}^i_{\ l} h_{jk} - \delta^i_{\ k} \widetilde{\phi}_{jl} + \delta^i_{\ l} \widetilde{\phi}_{jk} - \phi_{;r} \phi^{;r} \left(\delta^i_{\ k} h_{jl} - \delta^i_{\ l} h_{jk} \right), \tag{5}$$

where $\widetilde{\phi}_{ab} = \phi_{;ab} - \phi_{;a} \phi_{;b}$. Let d_h denote the distance function with respect to h and let d_g denote the distance function with respect to g. By [6, Theorem

1.8], there is a $\phi \in C^\infty(M)$ and a constant $c > 0$ such that
1. $\phi(m) \le d_h(m_0, m) \le \phi(m) + c$.
2. $\| \nabla\phi \|_\infty \le c$.
3. $\| \mathrm{Hess}(\phi) \|_\infty \le c$.

Then from (5), in order to show that g has quadratic curvature decay it suffices to show that there is a constant $C > 0$ such that $d_g(m_0, m) \le C\, e^{\phi(m)}$ for all $m \in M$. Let γ be a normalized minimal geodesic, with respect to h, from m_0 to m. Then measuring the length of γ with respect to g,

$$d_g(m_0, m) \le \int_0^{d_h(m_0,m)} e^{\phi(\gamma(t))} dt \le \int_0^{d_h(m_0,m)} e^t dt = e^{d_h(m_0,m)} - 1 \le e^c\, e^{\phi(m)}. \tag{6}$$

The lemma follows. Q.E.D.

3 Proof of Proposition 1.1

First of all, we show that every manifold with lower quadratic Ricci curvature decay has polynomial volume growth.

Lemma 3.1 *Suppose that there is a constant $C > 0$ such that for each $m \in M$ and each unit vector $v \in T_mM$, the Ricci curvature satisfies*

$$\mathrm{Ric}(v, v) \ge -(n-1)\frac{C}{d(m_0, m)^2}, \tag{7}$$

Put $N = (n-1)\,\frac{\sqrt{1+4C}-1}{2} + n$. Then there is a constant $C_0 = C_0(n, C) > 0$ such that for $t \ge 3$,

$$\mathrm{vol}(B_t) \le C_0 \mathrm{vol}(S_1)\, t^N + \mathrm{vol}(B_1) \tag{8}$$

and

$$\mathrm{vol}(B_{t+1} - B_{t-1}) \le C_0 \frac{\mathrm{vol}(B_{t-1})}{t-1}. \tag{9}$$

Proof: Let $\Pi_t = \frac{1}{n-1}\sum_{i=1}^{n-1} k_i$ denote the mean curvature of the regular part of S_t, where $\{k_i\}_{i=1}^{n-1}$ are the principal curvatures. Letting dA_t and dA_{m_0} denote the volume forms on S_t and $S_{m_0}M$ respectively, define $\varphi_t : S_{m_0}M \to S_t$ by

$$\varphi_t(v) = \exp_{m_0}(tv) \tag{10}$$

and define $\eta_t : S_{m_0}M \to (0,\infty)$ by

$$(\varphi_t)^* dA_t|_v = \eta_t(v)\, dA_{m_0}. \tag{11}$$

We have

$$\mathrm{vol}(S_t) = \int_{S_{m_0}M} \eta_t(v)\, dA_{m_0} \tag{12}$$

and

$$(n-1)\,\Pi_t|_{\varphi_t(v)} = \eta_t'(v)/\eta_t(v). \tag{13}$$

As $t \to 0$,

$$(n-1)\,\Pi_t|_{\varphi_t(v)} = \frac{n-1}{t} - \frac{\mathrm{Ric}(v,v)}{3} t + o(t). \tag{14}$$

Put $\Pi(t) = \Pi_t|_{\varphi_t(v)}$ and $v(t) = (\exp_{m_0})_*(tv)$. The Riccati equation implies

$$\Pi'(t) + \Pi(t)^2 \leq -\frac{\mathrm{Ric}(v(t),v(t))}{n-1}. \tag{15}$$

Put $\alpha = \frac{\sqrt{1+4C}+1}{2}$ and consider

$$f(t) = e^{\int_1^t \Pi(s)(s)ds}\Big[t^\alpha \Pi(t) - \alpha t^{\alpha-1}\Big]. \tag{16}$$

Then (14) implies that $\lim_{t\to 0+} f(t) = 0$. On the other hand, from (7) and (15), we have

$$f'(t) = t^\alpha\, e^{\int_1^t \Pi(s)(s)ds}\Big[\Pi'(t) + \Pi(t)^2 - \alpha(\alpha-1)t^{-2}\Big] \leq 0. \tag{17}$$

Thus $f(t) \leq 0$, giving

$$\Pi(t) \leq \alpha t^{-1}. \tag{18}$$

Together with (13), we conclude that $\eta_t(v)/t^{(n-1)\alpha}$ is nonincreasing. This implies that $\mathrm{vol}(S_t)/t^{(n-1)\alpha}$ is nonincreasing, too. As

$$\mathrm{vol}(B_t) - \mathrm{vol}(B_1) = \int_1^t \frac{\mathrm{vol}(S_s)}{s^{(n-1)\alpha}}\, s^{(n-1)\alpha}\, ds, \tag{19}$$

we obtain

$$\mathrm{vol}(S_1) \int_1^t s^{(n-1)\alpha}\, ds \;\geq\; \mathrm{vol}(B_t) - \mathrm{vol}(B_1) \;\geq\; \frac{\mathrm{vol}(S_t)}{t^{(n-1)\alpha}} \int_1^t s^{(n-1)\alpha}\, ds. \tag{20}$$

Hence

$$\mathrm{vol}(B_t) \;\leq\; \frac{1}{(n-1)\alpha+1}\,\mathrm{vol}(S_1)\,t^{(n-1)\alpha+1} + \mathrm{vol}(B_1). \tag{21}$$

Also,

$$\begin{aligned}
\mathrm{vol}(B_{t+1}-B_{t-1}) &= \int_{t-1}^{t+1} \frac{\mathrm{vol}(S_s)}{s^{(n-1)\alpha}}\, s^{(n-1)\alpha}\, ds \\
&\leq \frac{\mathrm{vol}(S_{t-1})}{(t-1)^{(n-1)\alpha}} \int_{t-1}^{t+1} s^{(n-1)\alpha}\, ds \\
&\leq \frac{\mathrm{vol}(B_{t-1})-\mathrm{vol}(B_1)}{\int_1^{t-1} s^{(n-1)\alpha} ds} \int_{t-1}^{t+1} s^{(n-1)\alpha}\, ds \\
&\leq C_0\, \frac{\mathrm{vol}(B_{t-1})}{t-1}
\end{aligned} \tag{22}$$

for large enough C_0. Q.E.D.

Proof of Proposition 1.1:
We use critical point theory of the distance function; for a review, see [3]. Let us say that a connected component Σ_t of S_t is *good* if it is part of the boundary of an unbounded component of $M - B_t$ and there is a ray from m_0 passing through Σ_t.

Lemma 3.2 *Suppose that there is a $t_0 > 0$ such that if $t \geq t_0$ then there is no critical point of d_{m_0} on any good component Σ_t of S_t. Then M has finite topological type.*

Proof: Let E be an end of M. We know that there is a normalized ray γ such that $\gamma(0) = m_0$ and γ exits E. By assumption, for all $t \geq t_0$, the connected component Σ_t of S_t which contains $\gamma(t)$ does not include any critical points of d_{m_0}. By the isotopy lemma [3, Lemma 1.4 and p. 35], it follows that the unbounded component U of $M - B_{t_0}$ containing $\{\gamma(t)\}_{t=t_0}^{\infty}$ is homeomorphic to $[0,\infty) \times \Sigma_{t_0}$, with Σ_{t_0} a closed connected topological manifold. In particular, $U \cap S_t$ is connected and good, so U does not contain any critical points. As S_{t_0} is compact, it has a finite number of connected components. It follows that $M - B_{t_0}$ has a finite number of bounded connected components and a finite number of unbounded connected components. Thus there is some $t_1 > t_0$ such that $M - B_{t_1}$ does not have any critical points, from which the lemma follows. In fact, the proof shows that M is homeomorphic

to the interior of a compact topological manifold-with-boundary. Q.E.D.

Define

$$\mathcal{D}(m_0,t)=\sup \mathrm{Diam}(\Sigma_t), \tag{23}$$

where the supremum is taken over all good components Σ_t of S_t and the diameter is measured using the metric on M. We claim that if the manifold has lower quadratic curvature decay and if

$$\lim_{t\to\infty}\frac{\mathcal{D}(m_0,t)}{t}=0 \tag{24}$$

there is a $t_0>0$ such that if $t\geq t_0$ then there is no critical point of d_{m_0} on any good component Σ_t of S_t.

For a pair of points $p,q\in M$, define

$$e_{pq}(x)=d(p,x)+d(q,x)-d(p,q).$$

Clearly, for any $t>0$ and any point $m\in M-B_{2t}$ on a ray from m_0 which intersects Σ_t,

$$e_{m_0m}(x)\leq 2\mathcal{D}(m_0,t)\quad \text{for } x\in\Sigma_t. \tag{25}$$

By assumption, the sectional curvature on $M-B_{t/2}$ satisfies

$$K_M\ \geq\ -\frac{4C}{t^2}. \tag{26}$$

Assume that there is a $t_0>0$ such that for $t>t_0$,

$$\mathcal{D}(m_0,t)\ \leq\ \frac{t}{4\lambda\sqrt{C}}, \tag{27}$$

where λ is a large constant which will be specified later.

Suppose that $x\in\Sigma_t$ is a critical point of d_{m_0}. Take a minimizing geodesic τ from x to m. There is a minimizing geodesic σ from x to m_0 such that $\angle(\dot\sigma(0),\dot\tau(0))\leq\frac{\pi}{2}$. Take two points $p=\sigma(a)$ and $q=\tau(a)$ where $a=\frac{t}{\lambda\sqrt{C}}$. By the triangle inequality, we have

$$e_{pq}(x)\leq e_{m_0m}(x)\leq 2\mathcal{D}(m_0,t). \tag{28}$$

For $\lambda\geq\frac{100}{\sqrt{C}}$, we see that the triangle Δ_{pxq} is contained in a small neighborhood of x inside $M-B_{t/2}$. Then we can apply the Toponogov inequality to Δ_{pxq} and obtain

$$\cosh(c_0d(p,q))\ \leq\ \cosh^2(c_0a), \tag{29}$$

where $c_0 = \frac{2\sqrt{C}}{t}$. Note that

$$c_0 d(p,q) = c_0[2a - e_{pq}(x)] \geq 2c_0[a - \mathcal{D}(m_0,t)] \geq \frac{3}{\lambda}. \tag{30}$$

We obtain

$$\cosh\left(\frac{3}{\lambda}\right) \leq \cosh^2\left(\frac{2}{\lambda}\right). \tag{31}$$

This is impossible for sufficiently large λ.

Finally, we must show that if $\mathrm{vol}(B_t) = o(t^2)$ and if there is a $v > 0$ such that $\mathrm{vol}(B_1(x)) > v$ for all $x \in M$, then (27) holds for large t.

Let Σ_t be a connected component of the boundary of an unbounded component of $M - B_t$. For any $x, y \in \Sigma_t$, there is a continuous curve $c : [0,r] \to \Sigma_t$ from x to y. Suppose that $d(x,y) > 2$. Then there is a partition $0 = t_0 < t_1 < \cdots < t_k = r$ such that $\{B_1(c(t_i))\}_{i=0}^k$ are disjoint and $B_2(c(t_i)) \cap B_2(c(t_{i+1})) \neq \emptyset$. Note that $B_1(c(t_i)) \subset B_{t+1} - \overline{B_{t-1}}$. We have

$$(k+1)v \leq \sum_{i=0}^{k} \mathrm{vol}(B_1(c(t_i))) \leq \mathrm{vol}(B_{t+1} - \overline{B_{t-1}}) \leq C_0 \frac{\mathrm{vol}(B_{t-1})}{t-1}. \tag{32}$$

Thus

$$\mathrm{Diam}(\Sigma_t) \leq \sum_{i=0}^{k-1} d(c(t_i), c(t_{i+1})) \leq C_1 \frac{\mathrm{vol}(B_{t-1})}{t-1}, \tag{33}$$

giving

$$\lim_{t\to\infty} \frac{\mathcal{D}(m_0,t)}{t} = 0. \tag{34}$$

This proves Proposition 1.1. Q.E.D.

4 Proof of Proposition 1.2

Fix an open set $\mathcal{O}$ containing E which is diffeomorphic to $(0,\infty) \times X$. For $u > 1$, let $\widehat{M}$ denote M with the metric $u^{-2}\, g_M$. Let $\widehat{\mathcal{O}}$ denote the copy of $\mathcal{O}$ in $\widehat{M}$. Let $\widehat{B}_t$ and $\widehat{S}_t$ denote the metric ball and metric sphere in $\widehat{M}$ around m_0. Rescaling (1), there is a constant $C' > 0$ such that the region $\widehat{B}_{100} - \widehat{B}_{1/100}$ has sectional curvatures bounded by C', uniformly in u. Put

$$T_{1/10}(\widehat{S}_1 \cap \widehat{\mathcal{O}}) = \{\widehat{m} \in \widehat{M} \;:\; d(\widehat{m}, \widehat{S}_1 \cap \widehat{\mathcal{O}}) \leq 1/10\}. \tag{35}$$

By [6, Theorem 0.1], there is a constant $C'' > 0$ independent of u such that there is a connected codimension-0 submanifold U_u of $\widehat{M}$ with

$$(\widehat{S}_1 \cap \widehat{\mathcal{O}}) \subset U_u \subset T_{1/10}(\widehat{S}_1 \cap \widehat{\mathcal{O}}), \tag{36}$$

$$\mathrm{vol}(\partial U_u) \leq C'' \,\mathrm{vol}(T_{1/10}(\widehat{S}_1 \cap \widehat{\mathcal{O}})) \tag{37}$$

and

$$\| \, \Pi_{\partial U_u} \, \| \leq \; C'', \tag{38}$$

where $\Pi_{\partial U_u}$ is the second fundamental form of ∂U_u in $\widehat{M}$. Then by the Gauss-Codazzi equation, the intrinsic sectional curvature of ∂U_u is uniformly bounded in u. Rescaling to M, we have

$$\mathrm{vol}(T_{1/10}(\widehat{S}_1 \cap \widehat{E})) = u^{-n} \,\mathrm{vol}(T_{u/10}(S_u \cap \mathcal{O})) \leq u^{-n} \,\mathrm{vol}(B_{11u/10}). \tag{39}$$

Let $\{u_j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{R}^+$ approaching infinity such that

$$\lim_{j \to \infty} \mathrm{vol}\left(B_{11u_j/10}\right)/u_j^n = 0. \tag{40}$$

For j large, let Y_j be a connected component of ∂U_{u_j}. Let $\mathcal{O}_j$ be the oriented cobordism between Y_j and X coming from the unbounded component of $M - Y_j$ corresponding to E, truncated at some level $\{R_j\} \times X$. Let $i : Y_j \to \mathcal{O}_j$ be the inclusion and let $\pi : \mathcal{O}_j \to (0, \infty) \times X \to X$ be projection. Then

$$\begin{aligned} &\int_X \omega \cup \prod_k p_{i_k}(TX) - \int_{Y_j} (\pi \circ i)^* \omega \cup \prod_k p_{i_k}(TY_j) = \\ &\int_{\mathcal{O}_j} d\left(\pi^* \omega \wedge \prod_k p_{i_k}(T\mathcal{O}_j)\right) = 0. \end{aligned} \tag{41}$$

From (37), (39), (40) and [9, p. 37], we have that $\int_{Y_j} (\pi \circ i)^* \omega \cup \prod_k p_{i_k}(TY_j) = 0$ if j is large enough. The proposition follows. Q.E.D.

5 Proof of Proposition 1.3

Suppose that $\{g(t)\}_{t \in [1,\infty)}$ is a smooth 1-parameter family of Riemannian metrics on X with sectional curvatures that are uniformly bounded in t. Then one can check that $dt^2 + t^2 \, g(t)$ is a metric of quadratic curvature decay on $[1, \infty) \times X$ if $\| \, g^{-1}(t) \, \frac{dg}{dt} \, \|_\infty = O\left(\frac{1}{t}\right)$ and $\| \, g^{-1}(t) \, \frac{d^2 g}{dt^2} \, \|_\infty = O\left(\frac{1}{t^2}\right)$. Put

$\delta = t^{-1}$ and let $g(t)$ be the Riemannian metric on X defined in [5, Section 3]. Then $\{g(t)\}_{t \in [1,\infty)}$ has uniformly bounded sectional curvature in t. We claim that $\parallel g^{-1}(t) \frac{dg}{dt} \parallel_\infty = O\left(\frac{1}{t}\right)$ and $\parallel g^{-1}(t) \frac{d^2g}{dt^2} \parallel_\infty = O\left(\frac{1}{t^2}\right)$. The metric $g(t)$ is defined by a finite recursive process. One starts with an invariant Riemannian metric g_0 for the F-structure and puts $g_1(t) = \log^2(1+t)\, g_0$. Clearly $\parallel g_1^{-1}(t) \frac{dg_1}{dt} \parallel_\infty = O\left(\frac{1}{t}\right)$ and $\parallel g_1^{-1}(t) \frac{d^2g_1}{dt^2} \parallel_\infty = O\left(\frac{1}{t^2}\right)$. Then

$$g_{j+1}(t) = \begin{cases} \rho_j^2\, g_j'(t) \; + \; h_j(t) & \text{on } U_j, \\ g_j(t) & \text{on } X - U_j. \end{cases} \tag{42}$$

where
1. U_j is a certain open subset of X,
2. $g_j'(t)$ is the part of $g_j(t)$ corresponding to tangent vectors to the F-structure on U_j,
3. $h_j(t)$ is the part of $g_j(t)$ corresponding to normal vectors to the F-structure on U_j and
4. $\rho_j = t^{-\frac{\log(f_j)}{\log(1/2)}}$ with $f_j : X \to [1/2, 1]$ a certain smooth function which is identically one on $X - U_j$.

It follows by induction on j that there is a metric of quadratic curvature decay and small volume growth on $[1,\infty) \times X$. Gluing $[1,\infty) \times X$ onto N, we obtain the desired metric on M. Q.E.D.

6 Proof of Proposition 1.4

If n is even then S^{n-1} has a polarized F-structure coming from a free S^1-action and the result follows from Proposition 1.3. The first nontrivial case is when $n = 3$.

Suppose that $n = 3$. By [4, Example 1.4], there is a metric h on $\mathbb{R}^3$ with finite volume and bounded sectional curvature. Our metric will be conformally related to h. Let us first give the construction of h in detail. For $j \in \mathbb{Z}^+$, let C_j be the complement of a small solid torus in a solid torus. Then topologically,

$$\mathbb{R}^3 = (S^1 \times D^2) \cup_{T^2} C_1 \cup_{T^2} C_2 \cup_{T^2} \ldots \tag{43}$$

We take $m_0 \in S^1 \times D^2$. Each C_j can be decomposed as $C_j = (\Sigma_{2j} \times S^1_{2j}) \cup_{T^2} (\Sigma_{2j+1} \times S^1_{2j+1})$, where Σ_{2j} is a 2-sphere with three disks removed, Σ_{2j+1} is

a 2-disk and S^1_{2j}, S^1_{2j+1} are circles. In [4, Figure 1.3], Σ_{2j} is represented as a rectangle with a disk removed and with the vertical sides identified. Put $\partial\Sigma_{2j} = S^1_{2j,1} \cup S^1_{2j,2} \cup S^1_{2j,3}$, where $S^1_{2j,1}$ is the top side of the rectangle, $S^1_{2j,2}$ is the bottom side of the rectangle and $S^1_{2j,3}$ is the circle enclosing the removed disk. Put $\partial\Sigma_{2j+1} = S^1_{2j+1,1}$. The identifications of the toroidal boundaries are

$$\begin{aligned} S^1_{2j+1,1} \times S^1_{2j+1} &\sim S^1_{2j,2} \times S^1_{2j}, \\ S^1_{2j,3} \times S^1_{2j} &\sim S^1_{2j-2,1} \times S^1_{2j-2}, \end{aligned} \tag{44}$$

where

$$\begin{aligned} S^1_{2j+1,1} &\sim S^1_{2j}, \\ S^1_{2j+1} &\sim S^1_{2j,2}, \\ S^1_{2j,3} &\sim S^1_{2j-2}, \\ S^1_{2j} &\sim S^1_{2j-2,1}. \end{aligned} \tag{45}$$

We will put product metrics on $\Sigma_{2j} \times S^1_{2j}$ and $\Sigma_{2j+1} \times S^1_{2j+1}$. Let ϵ_i be the length of S^1_i and let $\delta_{i,*}$ be the length of $S^1_{i,*}$. Then (45) gives the relations

$$\begin{aligned} \delta_{2j,1} &= \epsilon_{2j+2}, \\ \delta_{2j,2} &= \epsilon_{2j+1}, \\ \delta_{2j,3} &= \epsilon_{2j-2}, \\ \delta_{2j+1,1} &= \epsilon_{2j}. \end{aligned} \tag{46}$$

We will take $\epsilon_i = e^{-i}$. Let Σ_∞ be a thrice-punctured sphere with a Riemannian metric such that three ends $E_1, E_2, E_3 \cong (1, \infty) \times S^1$ are isometric to $dr^2 + e^{-2r}\, d\theta^2$. Put $\Sigma_0 = \Sigma_\infty - (E_1 \cup E_2 \cup E_3)$. Let $u \in C^\infty([0,1])$ be a nondecreasing function such that

$$\begin{cases} u(s) = s & \text{if } s \in [0, \frac{1}{3}], \\ 1 & \text{if } s \in [\frac{1}{2}, 1]. \end{cases} \tag{47}$$

Given $k \in \mathbb{Z}^+$, put $E(k) = [0,k] \times S^1$ with the metric $dr^2 + e^{-2ku(r/k)}\, d\theta^2$. Then put

$$\Sigma_{2j} = \Sigma_0 \cup_{\partial\Sigma_0} (E(2j+2) \cup E(2j+1) \cup E(2j-2)), \tag{48}$$

isometrically. Similarly, let Σ'_∞ be a once-punctured sphere with a Riemannian metric such that the end $E \cong (1,\infty) \times S^1$ is isometric to $dr^2 + e^{-2r}\, d\theta^2$. Put $\Sigma'_0 = \Sigma'_\infty - E$ and

$$\Sigma_{2j+1} = \Sigma'_0 \cup_{S^1} E(2j), \tag{49}$$

isometrically. Then one can check that $\{\Sigma_i\}_{i=1}^\infty$ have uniformly bounded volume and curvature. Glue together the product metrics on $\{\Sigma_{2j} \times S^1_{2j}\}_{j=1}^\infty$ and $\{\Sigma_{2j+1} \times S^1_{2j+1}\}_{j=1}^\infty$ to give the metric h on $\mathbb{R}^3$. As $\sum_{j=1}^\infty e^{-j} < \infty$, it follows that h has bounded curvature and finite volume.

Given $\phi \in C^\infty\left(\mathbb{R}^3\right)$, put $g = e^{2\phi} h$. By (5), the weighted sectional curvatures

$$\left\{e^{2\phi(m)}\, |K(P,g)|\right\}_{m\in M,\, P\subset T_m M} \tag{50}$$

are uniformly bounded provided that the gradient $\nabla\phi$ of ϕ and the Hessian $H(\phi)$ of ϕ are uniformly bounded with respect to h.

We construct ϕ on $\Sigma_{2j} \times S^1_{2j}$ and $\Sigma_{2j+1} \times S^1_{2j+1}$ to be the pullbacks of functions on Σ_{2j} and Σ_{2j+1}, respectively. Let $\phi_\infty \in C^\infty(\Sigma_\infty)$ be a Morse function with one critical point, of saddle type, such that

$$\begin{aligned} \phi_\infty|_{E_1} &= 40\, d(\cdot, \Sigma_0), \\ \phi_\infty|_{E_2} &= 10\, d(\cdot, \Sigma_0), \\ \phi_\infty|_{E_3} &= -\,80 \,-\, 40\, d(\cdot, \Sigma_0), \\ \phi_\infty(\Sigma_0) &\subset [-80, 0]. \end{aligned} \tag{51}$$

Then in terms of (48), put

$$\phi|_{\Sigma_{2j}} = 80j^2 + 80j + \phi_\infty|_{\Sigma_{2j}}. \tag{52}$$

Similarly, let $\phi'_\infty \in C^\infty(\Sigma'_\infty)$ be a Morse function with one critical point, a local maximum, such that

$$\begin{aligned} \phi'_\infty|_E &= -\,10\, d(\cdot, \Sigma'_0), \\ \phi'_\infty(\Sigma'_0) &\subset [0, 10]. \end{aligned} \tag{53}$$

Then in terms of (49), put

$$\phi|_{\Sigma_{2j+1}} = 80j^2 + 120j + 10 + \phi'_\infty|_{\Sigma_{2j+1}}. \tag{54}$$

Finally, define ϕ on the $S^1 \times D^2$ factor in (43) so as to extend ϕ to a smooth function on $\mathbb{R}^3$.

It is easy to see that $\nabla\phi$ and $H(\phi)$ are uniformly bounded on $\mathbb{R}^3$. As

$$d_g(m_0, m)^2 \, |K(P,g)| = \frac{d_g(m_0,m)^2}{e^{2\phi(m)}} \, e^{2\phi(m)} \, |K(P,g)|, \tag{55}$$

in order to show that g has quadratic curvature decay, it suffices to show that $e^{-\phi(m)} \, d_g(m_0, m)$ is uniformly bounded with respect to $m \in \mathbb{R}^3$. Let T^2 be the first torus factor in (43). Then it suffices to show that $e^{-\phi(m)} \, d_g(T^2, m)$ is uniformly bounded with respect to $m \in \mathbb{R}^3$. Let $\{\gamma(s)\}_{s \in [0,t]}$ be a piecewise smooth path from m to T^2 which is unit-speed with respect to h, and along which ϕ is nonincreasing. Then letting $L_g(\gamma)$ denote the length of γ with respect to g, we have

$$e^{-\phi(m)} \, d_g(T^2, m) \; \le \; e^{-\phi(m)} \, L_g(\gamma) = \int_0^t e^{\phi(\gamma(s)) - \phi(m)} \, ds. \tag{56}$$

We take γ to be (reparametrized) gradient flow of ϕ starting from m. Although ϕ is not a Morse function, we note that gradient flow on $\Sigma_{2j} \times S^1_{2j}$ is essentially the same as gradient flow on Σ_{2j}, as it is constant in the S^1_{2j}-factor, and gradient flow on $\Sigma_{2j+1} \times S^1_{2j+1}$ is essentially the same as gradient flow on Σ_{2j+1}, as it is constant in the S^1_{2j+1}-factor. If the projection of γ onto Σ_{2j} or Σ_{2j+1} meets a critical point c of saddlepoint type, we extend γ beyond c to become a piecewise smooth curve with a corner, again following a downward gradient trajectory. We continue this process until γ hits T^2. Changing variable to $u = \phi(m) - \phi(\gamma(s))$, we have

$$\int_0^t e^{\phi(\gamma(s)) - \phi(m)} \, ds = \int_0^{\phi(m)} e^{-u} \, \frac{du}{|\nabla\phi|(\phi^{-1}(u))}. \tag{57}$$

As $\phi(\gamma(s))$ is nonincreasing, if $m \in C_j$ then γ never enters $\Sigma_{2k+1} \times S^1_{2k+1}$ for $k < j$. Also γ hits at most one critical point in each Σ_{2k} for $k < j$. By the construction of ϕ, if $c_k \in \Sigma_{2k}$ is the critical point then $\phi\big|_{c_k \times S^1_{2k}} \in [80k^2 + 80k - 80, 80k^2 + 80k]$. Thus the singularities of $\frac{1}{|\nabla\phi|(\phi^{-1}(u))}$ are well-spaced in u. If γ passes through a critical point c and $u_0 = \phi(c)$ then $\frac{1}{|\nabla\phi|(\phi^{-1}(u))} \sim \frac{1}{\sqrt{|u - u_0|}}$ for $u \sim u_0$. From the uniform nature of $\nabla\phi$ near the critical points, it follows that there is a constant $D > 0$, independent of $m \in \mathbb{R}^3$, such that for all $x \in [0, \phi(m) - 1]$,

$$\int_x^{x+1} \frac{du}{|\nabla\phi|(\phi^{-1}(u))} \le D. \tag{58}$$

Then
$$\int_0^{\phi(m)} e^{-u} \, \frac{du}{|\nabla \phi|(\phi^{-1}(u))} \leq \frac{D}{1-e^{-1}}. \tag{59}$$
Thus g has quadratic curvature decay.

Put $t_{j+1} = d(m_0, C_{j+1})$. For $j > 0$, each path from m_0 to C_{j+1} must pass through C_j. Put
$$D_j = (S^1 \times D^2) \cup_{T^2} C_1 \cup_{T^2} \ldots \cup_{T^2} C_j. \tag{60}$$
Then $B_{t_{j+1}}(m_0) \subset D_j$ and so $\mathrm{vol}(B_{t_{j+1}}) \leq \mathrm{vol}(D_j)$. With respect to (48), let F_j be the subset $[j+2, 2j+2] \times S^1_{2j} \subset E(2j+2) \times S^1_{2j}$. For large j, $\phi|_{D_j - F_j} \leq 80j^2 + 120j + 80$ and so
$$\mathrm{vol}(D_j - F_j) \ \leq \ e^{240j^2+360j+240} \, \mathrm{vol}(\mathbb{R}^3, h). \tag{61}$$
On the other hand,
$$\mathrm{vol}(F_j) = \int_{j+2}^{2j+2} e^{3(80j^2+80j+40x)} e^{-2(2j+2)} dx = \frac{1-e^{-120j}}{120} e^{240j^2+480j+240} e^{-2(2j+2)}. \tag{62}$$
Thus
$$\mathrm{vol}(B_{t_{j+1}}) = O\left(e^{240j^2+480j+240} \, e^{-2(2j+2)}\right). \tag{63}$$
As any path from m_0 to C_{j+1} must pass through F_j,
$$t_{j+1} \ \geq \ \int_{j+2}^{2j+2} e^{80j^2+80j+40x} \, dx = \frac{1-e^{-40j}}{40} \, e^{80j^2+160j+80}. \tag{64}$$
Thus
$$\mathrm{vol}(B_{t_{j+1}})/t^3_{j+1} \ = \ O\left(e^{-2(2j+2)}\right), \tag{65}$$
showing that g has slow volume growth.

If $n > 3$, we can do a similar construction in which C_j is the complement of a small $T^{n-2} \times D^2$ in $T^{n-2} \times D^2$ and C_j is decomposed as $(\Sigma_{2j} \times T^{n-2}) \cup_{T^{n-1}} (\Sigma_{2j+1} \times T^{n-2})$. Q.E.D.

7 Proof of Corollary 1.1

1. If $n = 2$, put a metric on $\mathrm{Int}(N)$ with flat cylindrical ends.
2. If $n = 3$, suppose that ∂N consists of 2-spheres and 2-tori. For a 2-sphere

component of ∂N, put a metric coming from Proposition 1.4 on the corresponding end of $\mathrm{Int}(N)$. For a 2-torus component of ∂N, put a flat metric on the corresponding end $(1, \infty) \times T^2$ of $\mathrm{Int}(N)$. This gives the desired metric on $\mathrm{Int}(N)$. Now suppose that $\mathrm{Int}(N)$ has a metric with quadratic curvature decay and slow volume growth. From Proposition 1.2, the simplicial volume of ∂N must vanish. Thus ∂N consists of 2-spheres and 2-tori.
3. If $n = 4$, suppose that the connected components of ∂N are graph manifolds. Then ∂N has a polarized F-structure and Proposition 1.3 implies that there is a metric on $\mathrm{Int}(N)$ with quadratic curvature decay and slow volume growth. Now suppose that Thurston's Geometrization Conjecture holds and that $\mathrm{Int}(N)$ has a metric with quadratic curvature decay and slow volume growth. From Proposition 1.2, the simplicial volume of ∂N must vanish. From [10], this implies that the connected components of ∂N are graph manifolds. Q.E.D.

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